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# FAMILIES OF CUTS WITH THE MFMC-PROPERTY

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A family $\mathcal{F}$ of cuts of an undirected graph $G=(V, E)$ is known to have the weak MFMC-property if (i) $\mathcal{F}$ is the set of $T$-cuts for some $T \subseteq V$ with $|T|$ even, or (ii) $\mathcal{F}$ is the set of two-commodity cuts of $G$, i.e. cuts separating any two distinguished pairs of vertices of $G$, or (iii) $\mathcal{F}$ is the set of cuts induced (in a sense) by a ring of subsets of a set $T \subseteq V$. In the present work we consider a large class of families of cuts of complete graphs and prove that a family from this class has the MFMC-property if and only if it is one of (i), (ii), (iii).

## 1. Introduction

Let $\mathcal{F}$ be a family of nonempty subsets of a finite set $E$. For $l \in \mathbf{R}_+^E$, a map f: $\mathcal{F} \to \mathbf{R}_+$ is called *l-admissible* if the packing condition

$$\sum(f(F):\ e \in F \in \mathcal{F}) \leqq l(e)$$

holds for each $e \in E$ ($\mathbf{R}_+$ is the set of nonnegative reals). The maximum packing problem for $E$, $\mathcal{F}$ and $l$ is to find an $l$-admissible function $f$ on $\mathcal{F}$ with $1 \cdot f = \sum(f(F): F \in \mathcal{F})$ maximum; this maximum is denoted by $p(\mathcal{F}, l)$. We say that a subset $B \subseteq E$ *meets* $\mathcal{F}$ if $B$ meets every set $F$ in $\mathcal{F}$ (i.e. $B \cap F \neq \emptyset$). The *blocker* $b(\mathcal{F})$ of $\mathcal{F}$ is the collection of minimal (with respect to inclusion) subsets of $E$ meeting $\mathcal{F}$. It is easy to show that $p(\mathcal{F}, l) \leqq l(B)$ for each $B \in b(\mathcal{F})$ (for a real-valued function $g$ on a finite set $S$ and a subset $S' \subseteq S$, $g(S')$ denotes $\sum(g(x):\ x \in S')$). $\mathcal{F}$ is said to have the (weak) *MFMC-property* [8] if the equality

$$p(\mathcal{F}, l) = \min\{l(B): B \in b(\mathcal{F})\}$$

holds for any $l \in \mathbf{R}_+^E$. In other words, it means that all vertices of the unbounded polyhedron $\{x \in \mathbf{R}^E:\ x \geqq 0,\ x(F) \geqq 1 \forall F \in \mathcal{F}\}$ are integral.

The problem of characterizing (in "good" terms) the class of families having the MFMC-property is open and seems to be hard enough (if such a characterization exists at all). In the present work we consider a special class of families. Namely, we are interested in families of cuts of a complete undirected graph. For any finite set $W$, let $K_W$ denote the complete undirected graph with vertex-set and edge-set $W$ and $E(W)$, respectively. An edge between $x$ and $y$ may be denoted by $xy$. For $X \subseteq W$,

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$\partial X = \partial^W X$ denotes the set of edges of $K_W$ with one end in $X$ and the other in $W-X$. $C \subseteq E(W)$ is called a *cut* of $K_W$ if $C=\partial X$ for some proper subset $X$ of $W$, i.e.: $\emptyset \neq X \subset W$.

Throughout this paper we shall deal with the following objects: a complete graph $K_V$, a subset $T$ of $V$ and a collection $\mathscr{A}$ of proper subsets of $T$; we refer to $T$ and $\mathscr{A}$ as a set of *terminals* and a *scheme* on $T$, respectively. For a set $W \supseteq T$ and a scheme $\mathscr{A}$ on $T$, let $\mathscr{C}^W(\mathscr{A})$ denote the set of cuts $C$ in $K_W$ such that $C=\partial^W X$ for some $X \subset W$ with $X \cap T \in \mathscr{A}$. The question is: for what $V$, $T$ and $\mathscr{A}$ does $\mathscr{C}^V(\mathscr{A})$ have the MFMC-property? Here are three examples of such collections.

**Example 1.** $|T|$ is an even integer, and $\mathscr{A}$ consists of all odd subsets of $T$ (the members of $\mathscr{C}^V(\mathscr{A})$ are usually called $T$-cuts). It follows from a theorem of Edmonds and Johnson [2] that such a family $\mathscr{C}^V(\mathscr{A})$ has the MFMC-property (see also [7, 10]).

**Example 2.** $T=\{s, s', t, t'\}$ and $\mathscr{A}=\{\{s, t\}, \{s, t'\}\}$, i.e., $\mathscr{C}^V(\mathscr{A})$ consists of all cuts in $K_V$ containing both edges $ss'$ and $tt'$. The fact that $\mathscr{C}^V(\mathscr{A})$ has the MFMC-property follows from the two-commodity flow theorem of Hu [4] and a general theorem of Lehman [6] (see also [3, 9]).

**Example 3.** Let $\mathscr{A}$ be a ring of subsets of $T$ (i.e. $A, B \in \mathscr{A}$ implies $A \cap B, A \cup B \in \mathscr{A}$) with a minimal element $\{s\}$ and a maximal element $T-\{t\}$ for some $s, t \in T$. Then $\mathscr{C}^V(\mathscr{A})$ has the MFMC-property, which can be shown as follows. Let $l$: $E(V) \to \mathbf{R}_+$ be an arbitrary function. Consider the directed graph $H=(V, Z)$ in which $(x, y) \in Z$ if and only if $x \in A$ implies $y \in A$ for all $A \in \mathscr{A}$. By a well-known theorem in lattice theory (see [1]), a proper subset $A$ of $T$ is an element of $\mathscr{A}$ if and only if there is no edge in $H$ leaving $A$, i.e., having its tail in $A$ and its head in $T-A$. Let $G$ be the mixed multigraph $(V, E(V) \cup Z)$ with the set $E(V)$ of undirected edges and the set $Z$ of directed ones. Define the lengths $l'$ of the edges of $G$ by $l'(e)=l(e)$ if $e \in E(V)$ and $l'(e)=0$ if $e \in Z$, and let $\pi(x) (x \in V)$ be the length of a shortest path (with respect to $l'$) from $s$ to $x$ assuming that only paths whose direction corresponds to that of the edges in $Z$ are admitted. Let $\pi(s)=0=a_0<a_1<\ldots<a_m=\pi(t)$ be the different values of $\pi$ not exceeding $\pi(t)$. Put $X_i=\{x \in V: \pi(x)<a_i\}$, $f(\partial X_i)=a_i-a_{i-1}$ $(i=1, \ldots, m)$ and $f(C)=0$ for the remaining $C$'s in $\mathscr{C}^V(\mathscr{A})$. It is not difficult to show that (i): $f$ is $l$-admissible, (ii): if $U$ is the set of undirected edges of a shortest path in $G$ from $s$ to $t$, then $U$ meets $\mathscr{C}^V(\mathscr{A})$, and (iii): $1 \cdot f=\pi(t)=l(U)$, whence the result follows.

Now we shall introduce several definitions. Two schemes $\mathscr{A}$ and $\mathscr{A}'$ on the same set $T$ are called *equivalent* if $\mathscr{C}^T(\mathscr{A})=\mathscr{C}^T(\mathscr{A}')$ (and therefore $\mathscr{C}^V(\mathscr{A})=\mathscr{C}^V(\mathscr{A}')$ for any $V \supseteq T$). Clearly $\mathscr{A}$ and $\mathscr{A}'$ are equivalent if and only if, for any $A \subset T$, $\{A, T-A\} \cap \mathscr{A} \neq \emptyset$ implies $\{A, T-A\} \cap \mathscr{A}' \neq \emptyset$ and vice versa. We say that $\mathscr{A}'$ is an *odd*, *two-commodity* or *lattice* scheme if $\mathscr{A}'$ is equivalent to the scheme $\mathscr{A}$ in Example 1, 2 and 3, respectively. We say that a subset $X \subset W$ *separates* $Y, Z \subset W$ if one of $Y$ and $Z$ lies entirely in $X$ and the other one in $W-X$. Two terminals $s, z \in T$ are called *$\mathscr{A}$-equivalent* if there is no $A \in \mathscr{A}$ separating $\{s\}$ and $\{t\}$. A terminal $t \in T$ is called *$\mathscr{A}$-redundant* if for any $A \in \mathscr{A}$, there is a $D \in \mathscr{A}$ such that $s \in A^*$ and $D^*=A^*-\{s\}$ for some $A^* \in \{A, T-A\}$ and $D^* \in \{D, T-D\}$.

**Definition** A scheme $\mathscr{A}$ on $T$ is called *compact* if $T$ contains neither $\mathscr{A}$-equivalent pairs of terminals nor $\mathscr{A}$-redundant terminals.

It is clear that if $s \in T$ is $\mathscr{A}$-redundant, $T' = T - \{s\}$ and $\mathscr{A}' = \{A' \subset T' : A' = A - \{s\}$ for some $A \in \mathscr{A}\}$, then $\mathscr{C}^V(\mathscr{A}) = \mathscr{C}^V(\mathscr{A}')$ for any $V \supseteq T$. Also it is easy to show that if $s$ and $t$ are $\mathscr{A}$-equivalent terminals, $V' = V - \{s\}$, and $T'$ and $\mathscr{A}'$ are defined as above, then $\mathscr{C}^V(\mathscr{A})$ has the MFMC-property if and only if this property holds for $\mathscr{C}^{V'}(\mathscr{A}')$. Thus, without loss of generality, we may consider only compact schemes.

**Theorem 1.** *Let $|V| \geqq |T| + 2$, and let $\mathscr{A}$ be a compact scheme on $T$. Then the collection $\mathscr{C}^V(\mathscr{A})$ has the MFMC-property if and only if $\mathscr{A}$ is an odd or a two-commodity or a lattice scheme.*

In Section 4 the case $|V| = |T| + 1$ is investigated and we show that in this case only one new type of cut families having the MFMC-property is possible.

## 2. Proof of Theorem 1

The "if" part of Theorem 1 was discussed earlier, hence here we prove the "only if" part. Let $\mathscr{A}$ be a compact scheme on $T$. Without loss of generality, one may assume that $\mathscr{A}$ is a *symmetric* scheme, i.e. $A \in \mathscr{A}$ implies $T - A \in \mathscr{A}$.

Two subsets $A, A' \subset T$ are said to be *crossing* (in notation; $A \not\| A'$) if each of the four subsets $A \cap A'$, $A - A'$, $A' - A$ and $T - (A \cup A')$ is nonempty, and *laminar* (denoted by $A \| A'$) otherwise. A triple of pairwise crossing subsets will be called a *crossing triple*. We say that a pair $\{A, A'\}$ of members of $\mathscr{A}$ is *regular* if at least one of the following two possibilities holds : (i) $A - A'$, $A' - A \in \mathscr{A}$, (ii) $A \cap A'$, $A \cup A' \in \mathscr{A}$. In particular, $\{A, A'\}$ is regular if $A \| A'$ (since $\mathscr{A}$ is symmetric). For $l \in \mathbf{R}_+^{E(V)}$ and $x, y \in V$, $\mu_l(xy)$ denotes the distance between $x$ and $y$ in the graph $K_V$ with edge length function $l$. The proof of Theorem 1 is based on the following two lemmas, the former was proved in [5] while the proof of the latter one is contained in Section 3.

**Lemma 2.1.** *Let $\mathscr{A}$ be a scheme on $T$, let $V \supset T$, and let $\mathscr{C}^V(\mathscr{A})$ have the MFMC-property.*

(i) *If $B \in b(\mathscr{C}^V(\mathscr{A}))$ and $x \in V - T$, then $x$ is incident to an even numbers of edges in $B$.*
(ii) $p(\mathscr{C}^V(\mathscr{A}), l) = \min\{\sum(\mu_l(st) : st \in U)\}$,
*where the minimum is taken over all minimal subsets $U \subseteq E(T)$ meeting $\partial^T A$ for* every $A \in \mathscr{A}$, i.e., $U \in b(\mathscr{C}^T(\mathscr{A}))$. ▮

**Lemma 2.2.** *Let $\mathscr{A}$ be a scheme on $T$, let $V \supset T$, $|V| \geqq |T| + 2$, and let $\mathscr{C}^V(\mathscr{A})$ have the MFMC-property.*

(i) *If $A_1, A_2 \in \mathscr{A}$ are crossing and there is an $A_3 \in \mathscr{A}$ such that $A_3 \| A_1$ and $A_3 \| A_2$, then the pair $\{A_1, A_2\}$ is regular.*
(ii) *Each crossing triple in $\mathscr{A}$ contains at least one regular pair.*

Let $P = \{Y_1, \ldots, Y_p\}$ be a partition of $T$; we admit empty sets $Y_i$ in $P$. We say that $A \subset T$ *conforms* to $P$ if for every $i$ either $Y_i \subseteq A$ or $Y_i \cap A = \emptyset$. Let $(T_1, \ldots, T_k | Z_1, \ldots, Z_m)$ denote a partition $\{T_1, \ldots, T_k, Z_1, \ldots, Z_m\}$ of $T$ in which the sets $T_1, \ldots, T_k$ are distinguished.

**Definition.** A partition $(T_1, \dots, T_k|Z_1, \dots, Z_m)$ is called *essential* (with respect to $\mathscr{A}$) if (i) $k$ is an odd integer $\geqq 3$, (ii) each $T_i$ is nonempty ($Z_j$ may be empty), and (iii) for every $i=1, \dots, k$, there exists $A^i \in \mathscr{A}$ conforming to $\{T_1, \dots, T_k, Z_1, \dots, Z_m\}$ and such that $T_i \subseteq A^i$ and $T_j \cap A^i = \emptyset$ if $j \neq i$.

The following assertion is one of the main tools in our proof.

**2.3.** *Let* $|V| \geqq |T|+1$, *and let* $\mathscr{C}^V(\mathscr{A})$ *have the MFMC-property. Further let* $(T_1, \dots, T_k|Z_1, \dots, Z_m)$ *be an essential partition of* $T$, *and* $T'=T_1 \cup \dots \cup T_k$. *Then there exists* $A \in \mathscr{A}$ *conforming to* $\{T', Z_1, \dots, Z_m\}$. *In particular,* $T' \neq T$ *and if* $m=1$ *then* $T'$ (*and* $T-T'$) *belongs to* $\mathscr{A}$.

**Proof.** Choose a vertex $x \in V-T$ and a terminal $t_i$ in each $T_i$. Let $B$ be union of the sets $E(T_1), \dots, E(T_k)$, $E(Z_1), \dots, E(T_m)$ and $\{xt_i\colon i=1, \dots, k\}$. Two cases are possible. 1) $B$ does not meet $\mathscr{C}^V(\mathscr{A})$. Then there are $A \in \mathscr{A}$ and $X \subset V$ such that $X \cap T = A$ and $B \cap \partial X = \emptyset$. It follows from this that, for any $Y \in \{T', Z_1, \dots, Z_m\}$, either $Y \subseteq A$ or $Y \cap A = \emptyset$, as required. 2) $B$ meets $\mathscr{C}^V(\mathscr{A})$. Let $B'$ be a minimal subset in $B$ meeting $\mathscr{C}^V(\mathscr{A})$. By Lemma 2.1, $x$ is incident with an even number of edges in $B'$, therefore $xt_i \notin B'$ for some $i \in \{1, \dots, k\}$ since $k$ is odd. Let $A^i$ be the same as in the definition of the essential partitions. Then $B' \cap \partial^V A_i = \emptyset$, a contradiction. ∎

Now we turn to the proof of the theorem. Let $\mathscr{C}^V(\mathscr{A})$ have the MFMC-property (in fact, below we use only Lemma 2.2 and 2.3 rather than the condition $|V| \geqq |T|+2$). Let $\mathscr{A}^0$ be the collection of minimal sets in $\mathscr{A}$. Since $\mathscr{A}$ is symmetric, $|\mathscr{A}^0| \geqq 2$. The following two situations are possible: (i) each pair of members of $\mathscr{A}$ is regular, and (ii) there is at least one nonregular pair in $\mathscr{A}$.

First we study case (i). Observe that any $A \in \mathscr{A}^0$ and $D \in \mathscr{A}$ are laminar. For if some $A$ and $D$ are crossing, then, by the minimality of $A$, $A \cap D \notin \mathscr{A}$ and $A-D \notin \mathscr{A}$, contrary to the assumption that $\{A, D\}$ is regular. Thus, for any $A \in \mathscr{A}_0$ and $D \in \mathscr{A}$, we have either $A \subseteq D$ or $A \cap D = \emptyset$, and hence $|A|=1$ for any $A \in \mathscr{A}^0$, otherwise $T$ would contain $\mathscr{A}$-equivalent terminals. Let $\mathscr{A}^0 = \{\{s_i\}\colon i=1, \dots, k\}$ and $T' = \{s_i\colon i=1, \dots, k\}$. Consider three cases.

*Case* 1: $k=2$. Then each $A \in \mathscr{A}$ separates $\{s_1\}$ and $\{s_2\}$. Let $\mathscr{A}' = \{A \in \mathscr{A}\colon s_1 \in A\}$. We observe that $\mathscr{A}'$ is a ring of subsets of $T$ with minimal element $\{s_1\}$ and maximal element $T-\{s_2\}$. Indeed, for crossing $A, A' \in \mathscr{A}'$, the regularity of $\{A, A'\}$ and the fact that $A-A' \notin \mathscr{A}$ (since $s_1, s_2 \notin A-A'$) imply $A \cap A', A \cup A' \in \mathscr{A}'$.

*Case* 2: $k$ is odd. It is clear that the partition $P=(\{s_1\}, \dots, \{s_k\}|T-T')$ is essential. Applying (2.3) to $P$ we obtain that $T-T' \in \mathscr{A}$, contradicting the definition of $\mathscr{A}^0$.

*Case* 3: $k$ is an even integer $\geqq 4$. We show that $\mathscr{A}$ is the odd scheme on $T$.

(1) If $A \subset T'$ and $|A|$ is odd, then $A \in \mathscr{A}$. This follows from 2.3 applied to the essential partition $(\{s_1'\}, \dots, \{s_p'\}|T-A)$, where $A=\{s_1', \dots, s_p'\}$.

(2) If $A \subset T$ and $|A \cap T'|$ is even, then $A \notin \mathscr{A}$. Indeed, suppose that it is not so, and let $A \cap T' = \{s_1', \dots, s_p'\}$. Then the partition $(\{s_1'\}, \dots, \{s_p'\}, T-A|A-T')$ is essential, and by 2.3, $A-T' \in \mathscr{A}$; a contradiction with the definition of $\mathscr{A}^0$.

Thus, it remains to prove that $T'=T$. Suppose, for a contradiction, that $Q=T-T' \neq \emptyset$. We shall show that $A \cup Z \in \mathscr{A}$ for any $A \in \mathscr{A}$ and $Z \subseteq Q$, and therefore each terminal $s$ in $Q$ is $\mathscr{A}$-redundant, contrary to the compactness of $\mathscr{A}$.

(3) If $A\in\mathscr{A}$, $D\subset A\cap T'$ and $|D|$ is even, then $A-D\in\mathscr{A}$.
Indeed, let $W=(T'-A)\cup D$. Since $|T'|$ is even and $A\cap T'$ is odd (by (2)), then $|W|$ is odd, and hence, by (1), $W\in\mathscr{A}$. As the pair $\{A, W\}$ is regular, at least one of $A\cup W$ and $A-W=A-D$ is an $\mathscr{A}$. But $|(A\cup W)\cap T'|=|T'|$ is even, whence $A\cup W\notin\mathscr{A}$, by (2). Thus, $A-D\in\mathscr{A}$.

(4) If $A\in\mathscr{A}$, $D\subset T'-A$ and $|D|$ is even, then $A\cup D\in\mathscr{A}$.
For if $s\in A\cap T'$ and $W=D\cup\{s\}$, then $W\in\mathscr{A}$ and $W-A=D\notin\mathscr{A}$, whence $A\cup D=A\cup W\in\mathscr{A}$.

For $A\subset T'$ with $|A|$ odd, put $\mathscr{L}(A)=\{Z\subseteq Q\colon Z\cup A\in\mathscr{A}\}$. It follows from (3) and (4) that the set $\mathscr{L}=\mathscr{L}(A)$ does not depend on the choice of $A$. Let $Z, Z'\in\mathscr{L}(A)$. The regularity of $\{A\cup Z, A\cup Z'\}$ implies $A\cup Z\cup Z'$, $A\cup(Z\cap Z')\in\mathscr{A}$, and so $\mathscr{L}$ is a lattice. Furthermore, if $Z\in\mathscr{L}(A)$ then $Q-Z\in\mathscr{L}(T'-A)$ (since $\mathscr{A}$ is symmetric), i.e., $\mathscr{L}$ is a complemented lattice. This fact together with the condition that $T$ contains no pairs of $\mathscr{A}$-equivalent terminals implies $\mathscr{L}=2^Q$, as required. This completes Case 3 of (i).

Now we consider case (ii), i.e., when $\mathscr{A}$ contains a non-regular pair. We prove that $|T|=4$ and $\mathscr{A}$ is a two-commodity scheme on $T$. As it was pointed out earlier, for any $A\in\mathscr{A}^0$ and $D\in\mathscr{A}$, the pair $\{A, D\}$ is regular if and only if $A$ and $D$ are laminar. Furthermore, it follows from (i) in Lemma 2.2 that, for every $A\in\mathscr{A}$, there exists $D\in\mathscr{A}$ such that $A$ and $D$ are crossing. Let $T'=\cup(A\colon A\in\mathscr{A}^0)$. First we show that $|T'|=4$ and there is an ordering $s_1, s_2, s_3, s_4=s_0$ of the elements of $T'$ such that $\mathscr{A}^0=\{\{s_i, s_{i+1}\}\colon i=0, 1, 2, 3\}$. This splits into a number of simple assertions. Let $A$ be an arbitrary set in $\mathscr{A}^0$.

(1) There exist $A', A''\in\mathscr{A}^0$ such that $A'\cap A''=\emptyset$, $A\nparallel A'$ and $A\nparallel A''$.
Indeed, let $D\in\mathscr{A}$ be such that $A$ and $D$ are crossing, and let $A'$ be a minimal set in $\mathscr{A}$ contained in $D$. If $A'\| A$ then, by Lemma 2.2 (i) applied to $A$, $D$ and $A'$, the pair $\{A, D\}$ is regular, which is impossible. $A''$ is defined to be a minimal set in $\mathscr{A}$ contained in $T-D$.

(2) Let $A'\in\mathscr{A}^0$, $A\nparallel A'$, and let $D\in\mathscr{A}$. Then $D$ and $A''$ are crossing for exactly one $A''\in\{A, A'\}$.
Indeed, if $D\| A$ and $D\| A'$, then, by Lemma 2.2 (i), $\{A, A'\}$ is a regular pair. If $D\nparallel A$ and $D\nparallel A'$, then, by 2.2 (ii), $\{A, A', D\}$ contains a regular pair. In both cases we obtain a contradiction.

(3) If $A'\in\mathscr{A}^0$ and $A\nparallel A'$, then $|A\cap A'|=1$.
For supposing $s, t\in A\cap A'$, choose a set $D$ in $\mathscr{A}$ separating $\{s\}$ and $\{t\}$ (which exists because $\mathscr{A}$ is compact). The minimality of $A$ and $A'$ implies that $D\nparallel A$ and $D\nparallel A'$, which contradicts (2).

(4) Let $A', A''\in\mathscr{A}^0$, $A'\cap A''=\emptyset$, $A\cap A'=\{s'\}$ and $A\cap A''=\{s''\}$. Then $A=\{s', s''\}$.
Indeed, suppose, for a contradiction, that $W=A-\{s', s''\}$ is nonempty, and consider the essential partition $(A'-\{s'\}, A''-\{s''\}, W|\{s'\}, \{s''\}, Z)$, where $Z=T-(A\cup A'\cup A'')$. By 2.3, there must exist a set $D$ in $\mathscr{A}$ of one of the following types: 1) $\{s\}$, $s\in\{s', s''\}$, 2) $\{s', s''\}$, 3) $Z$, 4) $\{s\}\cup Z$, $s\in\{s', s''\}$, 5) $\{s', s''\}\cup Z$. Each of these case leads to a contradiction: if $D$ is as in 1 or 2, then $D\subset A$; if $D$ is as in 3, then $D\| A$ and $D\| A'$; if $D$ is as in 4 or 5, then $D\nparallel A$ and either $D\nparallel A'$ or $D\nparallel A''$.

Thus, we may assume that $\mathscr{A}^0$ contains $A_i=\{s_i, s_{i+1}\}$, $i=1, 2, 3$. Let $\tilde{A}$ be a set in $\mathscr{A}^0$ different from $A_i$, $i=1, 2, 3$ (such a set exists in view of (1)). By (4), $|\tilde{A}|=2$, and now using (2) we easily conclude that $\tilde{A}=A_0=\{s_4, s_1\}$, as required.

It remains to show that $Q=T-T'=\emptyset$. Suppose that it is not so. For any $D\in\mathscr{A}$, we obviously have $D\cap T'=A_i$ for some $i\in\{0,1,2,3\}$. Put $\mathscr{L}(A_i)=\{Z\subset Q\colon Z\cup A_i\in\mathscr{A}\}$. We show that $\mathscr{L}(A_i)=2^Q$, and therefore each terminal in $Q$ is $\mathscr{A}$-redundant, contrary to the compactness of $\mathscr{A}$. First observe that $\mathscr{L}(A_i)$ is a lattice. Indeed, if $D, D'\in\mathscr{A}$ are crossing and $D\cap T'=D'\cap T'=A_i$, then $\{D, D'\}$ is regular by Lemma 2.2 applied to $D$, $D'$ and $A_i$, whence $D\cap D'$, $D\cup D'\in\mathscr{A}$. Below all indices are taken modulo 4.

(5) If $Z\in\mathscr{L}(A_i)$ then $Q-Z\in\mathscr{L}(A_{i+1})$.
This is trivial if $Z=\emptyset$ or $Z=Q$. Otherwise consider the essential partition $(Z, \{s_i\}, \{s_{i-1}\}|\ \{s_{i+1}\}, \{s_{i+2}\}\cup(Q-Z))$ (One should choose $A^1$, $A^2$ and $A^3$ in the definition of the essential partitions to be $T-A_{i-1}$, $A_i$ and $T-(A_i\cup Z)$, respectively). By 2.3, there exists $D\in\mathscr{A}$ of one of the following types: 1) $\{s_{i+1}\}$, 2) $\{s_{i+2}\}\cup\cup(Q-Z)$, 3) $\{s_{i+1}, s_{i+2}\}\cup(Q-Z)=A_{i+1}\cup(Q-Z)$. Cases 1 and 2 are impossible because in the first case we have $D\subset A_i$ and in the second one $D\not\supset A_{i+1}$ and $D\not\supset A_{i+2}$. Hence $A_{i+1}\cup(Q-Z)\in\mathscr{A}$, as required.

Now (5) together with the obvious fact that $Z\in\mathscr{L}(A_i)$ if and only if $Q-Z\in\mathscr{L}(A_{i+2})$ implies that $\mathscr{L}=\mathscr{L}(A_i)$ does not depend on $i$ and $\mathscr{L}$ is a complemented lattice. Finally, using that $T$ has no pairs of $\mathscr{A}$-equivalent terminals we obtain $\mathscr{L}=2^Q$.

## 3. Proof of Lemma 2.2

The proof is based on the following lemma.

**Lemma 3.1.** *Let $|V|\geqq|T|+2$, let $\mathscr{A}$ be a scheme on $T$ such that $\mathscr{C}^V(\mathscr{A})$ has the MFMC-property, and let $S=\{A_1, A_2, A_3\}$ be a crossing triple in $\mathscr{A}$. Then there exists $D\in\mathscr{A}$ such that $D$ is laminar to at least two members of $S$.*

**Proof.** First of all we classify the crossing triples. For a crossing triple $Q=\{D_1, D_2, D_3\}\subset 2^W$, let $P(Q)$ denote the partition $\{W_1, \dots, W_m\}$ of $W$ into maximal subsets $W_j$ such that either $W_j\subseteq D_i$ or $W_j\cap D_i=\emptyset$ for $i=1,2,3$. We say that two triples $Q=\{D_1, D_2, D_3\}\subset 2^W$ and $Q'=\{D_1', D_2', D_3'\}\subset 2^{W'}$ with $P(Q)=\{W_1, \dots, W_m\}$ and $P(Q')=\{W_1', \dots, W_{m'}'\}$ are *equivalent* (denoted as $Q\sim Q'$) if $m=m'$ and there are a permutation $\sigma$ on $\{1,2,3\}$ and a permutation $\gamma$ on $\{1,\dots,m\}$ such that, for any $j,k\in\{1,\dots,m\}$ and $i\in\{1,2,3\}$, $D_i$ separates $W_j$ and $W_k$ if and only if $D_{\sigma(i)}$ separates $W_{\gamma(j)}$ and $W_{\gamma(k)}$. In Fig. 1 six pairwise non-equivalent crossing triples $S_1, \dots, S_6$ are illustrated. (Here, for $r=1,\dots,6$, every member of $P(S_r)$ consists of one element and, for each $D\in S_r$, a line separating $D$ and its complement is drawn). One can prove that an arbitrary crossing triple is equivalant to one of $S_1, \dots, S_6$. The proof reduces to a routine examination of various triples of subsets of 4, …, 8-element sets hence it is omitted.

Now suppose that the hypotheses of Lemma 3.1 hold. We shall use the following theorem due to Lehman [6]: a family $\mathscr{F}$ of subsets of a set $E$ has the MFMC-property if and only if the "length-width" inequality

$$(*)\qquad lw\geqq\min\{l(B)\colon B\in b(\mathscr{F})\}\min\{w(F)\colon F\in\mathscr{F}\}$$

holds for any $l\in\mathbf{R}_+^E$ and $w\in\mathbf{R}_+^E$, where $b(\mathscr{F})$ is the blocker of $\mathscr{F}$ and $lw=\sum(l(e)w(e)\colon e\in E)$. We apply the "only if" part of this theorem to our $E=E(V)$

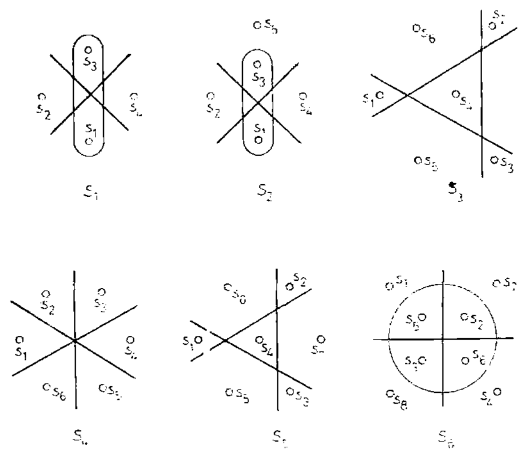

*Fig. 1*

and $\mathscr{F}=\mathscr{C}^V(\mathscr{A})$. Using (ii) of Lemma 2.1, one can replace $(*)$ by

$$(1)\qquad lw \geqq \min\Big\{\sum_{st\in U}\mu_l(st):\ U\in b(\mathscr{C}^T(\mathscr{A}))\Big\}\min\{w(C):\ C\in\mathscr{C}^V(\mathscr{A})\}.$$

For $l\in\mathbf{R}_+^E$, let

$$v(l,S)=\min\Big\{\sum_{st\in Y}\mu_l(st):\ Y\in b(\mathscr{C}^T(S))\Big\}.$$

Since every $U\in b(\mathscr{C}^T(\mathscr{A}))$ meets the family $\mathscr{C}^T(S)=\{\partial^T A_i: i=1,2,3\}$, then

$$(2)\qquad v(l,S)\leqq\min\Big\{\sum_{st\in U}\mu_l(st):\ U\in b(\mathscr{C}^T(\mathscr{A}))\Big\}.$$

For $w\in\mathbf{R}_+^{E(V)}$ and $A\subset T$, define

$$d(w,A)=\min\{w(\partial^V X):\ X\subset V,\ X\cap T=A\}$$

and

$$a(w,S)=\min\{d(w,A_i):\ i=1,2,3\}.$$

Let $\mathscr{R}(w)$ denote the collection of proper subsets $A$ of $T$ such that $d(w,A)<a(w,S)$.

The idea of the proof is as follows. We shall find functions $l, w\in\mathbf{R}_+^{E(V)}$ satisfying

$$(3)\qquad lw<v(l,S)a(w,S).$$

So it will follow from (1), (2) and (3) that $\mathscr{A}\cap\mathscr{R}(w)\neq\emptyset$. A function $w$ will be chosen so that each $D\in\mathscr{R}(w)$ is laminar to at least two members of $S$, whence the result will follow.

Let $P(S)=\{T_1,\ldots,T_m\}$; we shall assume that if $S\sim S_r$ then the indices in $P(S)$ correspond to ones in $P(S_r)$ as shown in Fig. 1 (That is the permutation $\gamma$ for given $S$ and $S_r$ is identical). Take one element $s_j$ in each $T_j$ (it is convenient to use the same symbols as in Fig. 1).

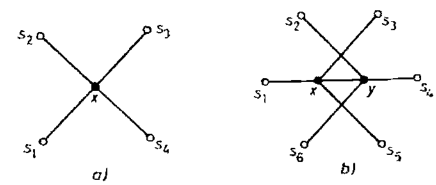


*Fig. 2*

(a) If $S$ is equivalent to one of $S_1, S_2, S_3, S_5, S_6$, we choose a vertex $x \in V-T$ and fix the subgraph $G'=(V', E')$ in $K_V$ with the vertex-set $V'=\{x, s_1, s_2, s_3, s_4\}$ and the edge-set $E'=\{s_i x\colon i=1, \dots, 4\}$ (see Fig. 2a).

(b) If $S \sim S_4$ we choose two vertices $x, y \in V-T$ and fix the subgraph $G'=(V', E')$ with vertex-set $V'=\{x, y, s_1, \dots, s_6\}$ and edge-set as illustrated in Fig. 2b. (This is the only place of our proof where we use the condition $|V| \geqq |T|+2$).

Define two functions $l$ and $w'$ on $E(V)$ by

$$l(e)=\begin{cases} 1 & \text{if } e \in E', \\ |\{i\colon 1 \leqq i \leqq 3, \ e \in \partial^T A_i\}| & \text{if } e \in E(T), \\ M & \text{otherwise,} \end{cases}$$

$$w'(e)=\begin{cases} 1 & \text{if } e \in E', \\ M & \text{if } e \in E(T_j), \quad j=1, \dots, m, \\ 0 & \text{otherwise,} \end{cases}$$

where $M$ is a large positive number. One can check that: (i) if $S$ is equivalent to one of $S_1, S_2, S_3, S_5, S_6$ then $v(l, S) \geqq 3$, $a(w', S)=2$ and $lw'=|E'|=4$; and (ii) if $S \sim S_4$ then $v(l, S)=3$, $a(w', S)=3$ and $lw'=|E'|=7$. (The proof is left to the reader; see also [5].) Thus, in each case we have $lw' < v(l, S)\, a(w', S)$. Note also that the inequality $w'(\partial^V X) < M$ holds only if the set $D = X \cap T$ conforms to $P(S)$, and hence $\mathscr{R}(w')$ can contain only such sets $D$. If $S \sim S_1$ then $P(S)=\{T_1, T_2, T_3, T_4\}$ and, obviously, no members distinct from $T_j$ (and $T-T_j$) can be in $\mathscr{R}(w')$; and so, our lemma is valid in this case. (Below we shall see that this case cannot occur, but at this point it does not matter.) Unfortunately, in the other five cases $\mathscr{R}(w')$ is still too large and this does not enable us to take $w'$ as required. For this reason, we define $w$ to be $w'+w''$, where $w'' \in \mathbf{R}_+^{E(V)}$ is a function satisfying

$$w''(\partial A_i) = q, \quad i = 1, 2, 3, \tag{4}$$

$$w''(e) = 0 \quad \text{for each} \quad e \in E(V) - E(T), \tag{5}$$

and such that

$$d(w', D) + w''(\partial^T D) \geqq a(w', S) + q \tag{6}$$

will be valid for each "undesirable" $D$ in $\mathscr{R}(w')$, i.e., if $|\{i\colon D \| A_i\}| \leqq 1$, where $q$ is a positive number. Note that (5) implies $w''(\partial^V X) = w''(\partial^T A)$ for any $X \subset V$ and $A = X \cap T$. Thus, (6) can be rewritten as $d(w, D) \geqq a(w, S)$, and therefore $D$ does not belong to $\mathscr{R}(w)$. Next, by (4), (5) and the definition of $l$, we have

$$lw'' = \sum \big(l(e)w''(e)\colon\ e\in E(T)\big) = \sum \big(w''(\partial^T A_i)\colon\ i = 1, 2, 3\big) =$$
$$= 3q \leqq v(l, S)a(w'', S),$$

and so, (3) is true for any considered $l$ and $w$.

It remains to point out such functions $w''$ for each of cases $S\sim S_r$, $r=2, \dots, 6$. These functions are illustrated in Fig. 3. Here multigraphs $H_2, \dots, H_6$ are drawn;

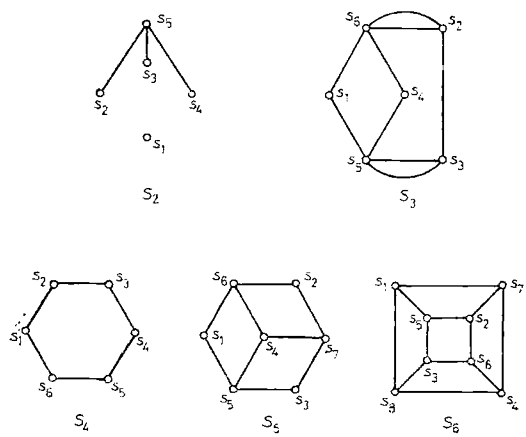


*Fig. 3*

for $S\sim S_r$ and $xy\in E(T)$, $w''(xy)$ is defined to be the number of edges of $H_r$ connecting the vertices $x$ and $y$. It is easy to check that (4) is true for any $w''$. The verification of (6) requires examination of subsets $D$ of form $T_j\cup \dots \cup T_{j'}$ which are crossing with at least two members of $S$. For example, let $S\sim S_2$. (We consider this case because we shall need it again in what follows). We have $a(w', S)=2$ and $q=1$. One can see that $\mathscr{R}(w')$ consists of $\{T_1\}, \dots, \{T_5\}, \{T_5, T_j\}$, $j=1, \dots, 4$, and their complementary sets. But for $D=\{T_5\}$, $\{T_5, T_j\}$, $j=1, \dots, 4$, the left hand side of (6) is at least 3, thus $\mathscr{R}(w)$ consists only of $\{T_1\}, \dots, \{T_4\}$ and their complements, each of which is laminar to all $A_i$, $i=1, 2, 3$. The other cases can be checked similarly. This completes the proof of Lemma 3.1. ∎

Using Lemma 3.1, we now prove Lemma 2.2.

(1) If $A, A', D\in\mathscr{A}$, $A\not\parallel A'$ and $D=T-(A\cup A')$, then $A\cap A'\in\mathscr{A}$. This follows from 2.3 applied to the essential partition $(A-A', A'-A, D|A\cap A')$.

(2) Let $A, A'\in\mathscr{A}$ be crossing. Then $A\Delta A'\notin\mathscr{A}$, where $\Delta$ denotes symmetric difference.

Indeed, suppose, for a contradiction, that $A\Delta A'\in\mathscr{A}$. Clearly, both $(A-A', A'-A, T-(A\cup A')|A\cap A')$ and $(T-(A\cup A'), A\cap A', A'-A|A-A')$ are essential partitions, whence, by (2.3), $A\cap A'\in\mathscr{A}$ and $A-A'\in\mathscr{A}$. But then the partition $(A\cap A', A-A', T-A|\emptyset)$ must be also essential, a contradiction.

(3) Let $A, A', D\in\mathscr{A}$, $A\not\parallel A'$, $D\subset T-(A\cup A')$, $Z=A\cap A'$ and $Z'=$ $=T-(A\cup A'\cup D)$. Then at least one of $Z, A-A', A'-A$ and $Z\cup Z'$ is a member of $\mathscr{A}$.

Indeed, applying 2.3 to the essential partition $(A-A', A'-A, D|Z, Z')$ shows that one of $Z, Z'$ and $Z\cup Z'$ is in $\mathscr{A}$. Supposing $Z'\in\mathscr{A}$ and applying 2.3 to $(D, Z', Z|A-A', A'-A)$, we obtain that one of $A-A'$, $A'-A$ and $A\Delta A'$ is in $\mathscr{A}$. Now the result follows from (2).

Next we prove (i) from Lemma 2.2. Let $A_1, A_2, A_3\in\mathscr{A}$, $A_1 \not\parallel A_2$ and $A_3\| A_i$, $i=1, 2$. One may assume that $A_3\subseteq T-(A_1\cup A_2)$. In view of (1), it suffices to show that one of $A_1-A_2$, $A_2-A_1$, $A_1\cap A_2$ and $T-(A_1\cup A_2)$ is a member of $\mathscr{A}$. Thus, we may assume that $A_3$ does not coincide with $T-(A_1\cup A_2)$. Let $Z=A_1\cap A_2$ and $Z'=T-(A_1\cup A_2\cup A_3)$. By (3), if none of $Z, A_1-A_2$ and $A_2-A_1$ is in $\mathscr{A}$, then $Z\cup Z'\in\mathscr{A}$. Clearly $S'=\{A_1, A_2, Z\cup Z'\}$ is a crossing triple and $S'\sim S_2$. It was shown in the proof of Lemma 3.1 that in this case at least one of $A_1-A_2$, $A_2-A_1$, $Z$ and $Z'$ is in $\mathscr{A}$. But if $Z'\in\mathscr{A}$, then, by (1) (used with $A=A_1$, $A'=T-(Z\cup Z')$ and $D=Z'$), we have $A_1\cap(T-(Z\cup Z'))=A_1-A_2\in\mathscr{A}$. This proves part (i) of Lemma 2.2.

Lemma 2.2 (ii) follows immediately from (i) and Lemma 3.1. This completes the proof of Lemma 2.2 and Theorem 1.

## 4. The case $|V|=|T|+1$

If we weaken the hypothesis of Theorem 1 by setting $|V|\geqq|T|+1$ instead of $|V|\geqq|T|+2$, then a priori schemes $\mathscr{A}$ different from the ones described in the theorem may appear, for which $\mathscr{C}^V(\mathscr{A})$ has the MFMC-property. Let $T=\{s_1, \dots, s_6\}$, and let $\mathscr{A}_6^4$ denote the scheme on $T$ consisting of four subsets $A_i=\{s_i, s_{i+1}, s_{i+2}\}$, $i=1, 2, 3$, and $A_4=\{s_1, s_3, s_5\}$. We say that a scheme $\mathscr{A}$ on $T$ is *octahedral* if it is equivalent to $\mathscr{A}_6^4$. (A motivation of using such a term is that if $|V|=|T|+1$ and $\{x\}=V-T$ and if the elements of $T$ are thought to be the vertices of an octahedron $\mathscr{O}$ and $x$ to be its centre, then the cuts of $\mathscr{C}^V(\mathscr{A})$ correspond to the eight partitions of $V$ induced by planes parallel to faces of $\mathscr{O}$.)

**Theorem 2.** *Let* $|V|=|T|+1$, *let* $\mathscr{A}$ *be a compact scheme on* $T$, *and let* $\mathscr{A}$ *be not octahedral* (*if* $|T|=6$). *Then* $\mathscr{C}^V(\mathscr{A})$ *has the MFMC-property if and only if* $\mathscr{A}$ *is an odd or a two-commodity or a lattice scheme.*

A sketch of a proof is as follows. Let $|V|=|T|+1$, and $\mathscr{A}$ be a symmetric compact scheme on $T$ having the MFMC-property. One can see from the proof of Theorem 1 that if both statements in Lemma 2.2 hold for some $\mathscr{A}$ and $V$, then $\mathscr{A}$ is an odd or a two-commodity or a lattice scheme (since in the proof developed in Section 2 we use, in fact, the condition $V\supset T$ rather than $|V|\geqq|T|+2$). Furthermore, one can see from the proofs of Lemmas 2.2 and 3.1 that part (i) of Lemma 2.2 as well as part (ii) in the cases $S\not\sim S_4$ remain true for given $\mathscr{A}$ and $V$. Thus, we may assume that $\mathscr{A}$ contains a triple $S'=\{A_1, A_2, A_3\}$ such that $S'\sim S_4$, $S'$ contains no regular pair and there exists no $D\in\mathscr{A}$ laminar with at least two members of $S'$. We show then that $\mathscr{A}$ is octahedral. We may assume also that $A_i=T_i\cup T_{i+1}\cup T_{i+2}$, $i=1, 2, 3$, where $\{T_1, \dots, T_5, T_6=T_0\}$ is a partition of $T$ into nonempty subsets. We observe that $T_j\notin\mathscr{A}$ and $T_j\cup T_{j+1}\notin\mathscr{A}$ for $j=0, \dots, 5$ (otherwise $S'$ would contain a regular pair), and that $T_j\cup T_{j'}\notin\mathscr{A}$ if $2\leqq|j'-j|\leqq 4$ (otherwise $\{A_1, A_2, A_3, T_j\cup T_{j'}\}$ would contain a crossing triple equivalent to $S_3$, which implies $T_j \in \mathscr{A}$ or $T_{j'} \in \mathscr{A}$). Now applying 2.3 to the essential partition $(T_2, T_4, T_6 | T_1, T_3, T_5)$ shows that $A_4 = T_1 \cup T_3 \cup T_5 \in \mathscr{A}$. Clearly, each triple of $A_1, A_2, A_3, A_4$ is equivalent to $S_4$ and each pair is non-regular. Let $\mathscr{A}'$ be the set of members of $\mathscr{A}$ different from $A_1, A_2, A_3, A_4$ and their complements with respect to $T$. We show that $\mathscr{A}' = \emptyset$.

Supposing the opposite, let $D$ be a member of $\mathscr{A}'$ with $|D|$ minimal. Let $\mathscr{R}$ be the set of $A_i$'s $(1 \leqq i \leqq 4)$ such that $D$ and $A_i$ are crossing. Clearly $|\mathscr{R}| = 3$ or 4. The minimality of $D$ implies that $\{D, A_i\}$ is non-regular for every $A_i \in \mathscr{R}$, and therefore each triple in $\mathscr{R} \cup \{D\}$ is equivalent to $S_4$. Let, for definiteness, $A_1, A_2, A_3 \in \mathscr{R}$. It is easy to show that $T_j \cup T_{j+1} \subset D \subset T_{j-1} \cup T_j \cup T_{j+1} \cup T_{j+2}$ for some $0 \leqq j \leqq 5$ and $D \cap T_k \neq \emptyset$, $T_k$ for $k = j-1$, $j+2$ (indices are taken modulo 5), which implies $D \not\parallel A_4$. Thus, $|\mathscr{R}| = 4$. Now, assuming, e.g. that $T_1 \cup T_2 \subset D$ we obtain $\{A_2, A_4, D\} \sim \sim S_6$, a contradiction. Finally, since $T$ has no $\mathscr{A}$-equivalent pairs, $|T_j| = 1$ for $j = 1, \ldots, 6$. Therefore, $\mathscr{A}$ is an octahedral scheme, as desired. ▮

Unfortunately, I do not know whether or not the family $\mathscr{C}^V(\mathscr{A}_6^4)$, $(|V| = 7)$ has the MFMC-property, but I think it has.

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